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# VALUATION RINGS IN FUNCTION FIELDS ONTO LATTICES

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**Abstract:** Taking a complete Heyting algebra $L$ and using $L$-sets, we will build the $L$-subrings of valuation and $L$-valuations of an algebraic function field of one variable $F/K$, as a generalization of the valuation rings and discrete valuations of $F/K$, and we will obtain many properties of them, and their analogues to the Theorem of Approximation of an amount finite of non-equivalent valuations.

**Keywords and Phrases:** Function fields, Discrete valuations, Ring valuations, Lattices, Fuzzy sets, Fuzzy rings, $L$-set, $L$-subrings.

**2020 Mathematics Subject Classification:** 13F30, 11T06, 11H06.

## 1. Introduction

The concept of fuzzy set was introduced in the year of 1965 by Lotfi Asker Zadeh in his paper entitled "Fuzzy Sets" (see [7]), in which he offers, in a certain direction, generalizations of some basic concepts of the set algebra. Three years later, C. L. Chang applied the concept of fuzzy set to realize many generalizations for the concepts of general topology (see [2]). Since then, several people have applied the concept of fuzzy set, or some concept of set more general, in various branches of mathematics and physics to obtain generalizations in this sense. We note that the fuzzy sets are defined using the interval $[0, 1]$ but, in this work, we will use an $L$ complete Heyting algebra to apply the $L$-sets (see [3]). With them, we will obtain generalizations of the valuation rings (Definition 2.1) and discrete valuations (Definition 4.1) of algebraic function field of one variable over a field, and we will obtain the counterpart theorems for $L$-valuations corresponding to the Theorem of Approximation of an amount finite non-equivalent valuations (Theorem 3.1 and Theorem 4.2).

So, we will fix the notation, considering in the first instance the one established in [4]. Let $K$ be a field fixed. An *algebraic function field of one variable over* $K$ is a field extension $F/K$ such that there exist $x \in F$ transcendental over $K$ where the extension $F/K(x)$ is finite. For simplicity, we will say that $F/K$ is a *function field*. We will denote by $\tilde{K}$ to the set of the elements $z \in F$ which are algebraic over $K$; it is a subfield of $F$ containing $K$ and is called the *field of constants* of $F/K$. A *valuation ring* $\mathcal{O}$ of $F/K$ is a subring of $F$ so that $K \subsetneqq \mathcal{O} \subsetneqq F$ and, for each element $z \in F$ nonzero, it is true that $z \in \mathcal{O}$ or $z^{-1} \in \mathcal{O}$. For a valuation ring $\mathcal{O}$ of $F/K$ is has that $\mathcal{O}$ is a local ring with a unique maximal ideal $P = \mathcal{O} \setminus \mathcal{O}^*$ where $\mathcal{O}^*$ is the multiplicative group of the units of the valuation ring $\mathcal{O}$, $\tilde{K} \subseteq \mathcal{O}$ and $\tilde{K} \cap P = \{0\}$. Furthermore, $\mathcal{O}$ is a principal ideal domain and if $t \in \mathcal{O}$ such that $P = t\mathcal{O}$, then each element $z \in F$ non zero is uniquely expressed of the form $z = ut^n$ for some $u \in \mathcal{O}^*$ and $n \in \mathbb{Z}$; such element $t \in \mathcal{O}$ is called *prime element* or *uniformizing element* for $P$. In these conditions, $\mathcal{O}_P := \mathcal{O}$ is called the *valuation ring of the place* $P$.

Now, as in [3], let $L$ be a order partially ordered set. We say that $L$ is a *complete Heyting algebra* if $L$ is a complete lattice such that for all $A \subseteq L$ and for each $a \in L$, $\sup\{a \wedge c \mid c \in A\} = a \wedge \sup A$ and $\inf\{a \vee c \mid c \in A\} = a \vee \inf A$. We will always assume that $L$ is a complete Heyting algebra consisting of at least three elements, where we write 1 and 0 for the maximal and minimal elements of $L$, and we will simply say that $L$ is a *lattice* as abbreviation. We say that $L$ is *regular* if for each $a, b \in L$ nonzero, $a \wedge b \neq 0$. If $X$ is a nonempty set, an *$L$-subset* of $X$ is any function from $X$ into $L$. In particular, when $L = [0, 1]$, the $L$-subsets of $X$ are called *fuzzy subsets*. If $\mu : X \longrightarrow L$ is an $L$-subset of $X$, the *image* of $\mu$ is the subset $\mu(X) := \{\mu(x) \mid x \in X\}$ of $L$ and the *support* of $\mu$ is the subset $\mu^* := \{x \in X \mid \mu(x) > 0\}$ of $X$. For all $a \in L$, the subset $\mu_a := \{x \in X \mid \mu(x) \geq a\}$ of $X$ is called the *$a$-cut* or *$a$-level set* of $\mu$. We say that $\mu$ is *normal* or *unitary* $L$-subset if $1 \in \mu(X)$. If $\nu$ is other $L$-subset of $X$, then we say that $\mu$ *is contained* *in* $\nu$ or $\nu$ *contains* $\mu$, and we write $\mu \subseteq \nu$ or $\nu \supseteq \mu$, if $\mu(x) \leq \nu(x)$ for all $x \in X$. On the other hand, if $R$ is a commutative ring and $\mu$ is an $L$-subset of $R$, we say that $\mu$ is an $L$*-subring* of $R$ if $\mu(x-y) \geq \mu(x) \wedge \mu(y)$ and $\mu(xy) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in R$; if in addition, $\mu$ satisfies the relation $\mu(xy) \geq \mu(x) \vee \mu(y)$ for all $x, y \in R$, we say $\mu$ is an $L$*-ideal* of $R$ and $\mu_* := \{x \in R \mid \mu(x) = \mu(0)\}$.

## *L*-subrings of valuations of function fields

**Definition 2.1.** *Let $F/K$ be a function field. An $L$-subring $\nu$ of $F$ is said to be* ***L-subring of valuation of*** *$F$ if there exists $a \in L \setminus \{0,1\}$ so that*

(*i*) *for each $z \in F$, $a$ and $\nu(z)$ are comparable;*

(*ii*) *for all $c \in K \setminus \{0\}$, $\nu(c) = a$;*

(*iii*) *there exists an element $z$ in $F \setminus \{0\}$ such that $\nu(z) > a$;*

(*iv*) *for each $z \in F \setminus \{0\}$, $\nu(z) > a$ if and only if $\nu(z^{-1}) < a$.*

The element $a \in L \setminus \{0,1\}$ of the Definition 2.1, is referred as the **divisor element** of $\nu$.

**Proposition 2.1.** *Let $\nu$ be an $L$-subring of valuation of a function field $F/K$. Then,*

(*i*) *$\nu(0) > a$;*

(*ii*) *$K \subsetneq \nu_a \subsetneq F$;*

(*iii*) *for each $z \in F \setminus \{0\}$, $\nu(z) \geq a$ or $\nu(z^{-1}) \geq a$;*

(*iv*) *$\nu_a$ is a valuation subring of $F$;*

(*v*) *$\tilde{K} \subseteq \nu_a$. Moreover, for each $z \in \tilde{K}$ with $z \neq 0$, $\nu(z) = a$ .*

(*vi*) *Let $u \in F \setminus \{0\}$. Then, $\nu(u) = a$ if and only if $u$ is a unity of $\nu_a$;*

(*vii*) *if $z$ and $w$ are elements of $F$ such that $\nu(z) > a$ and $\nu(w) \geq a$, then $\nu(zw) > a$. In particular, if $\nu(z) > a$ and $\nu(w) = a$, then $\nu(zw) > a$.*

(*viii*) *Let $t \in F$ be a prime element of the place associated to the valuation ring $\nu_a$ of $F$. Then, $\nu(t^n) \geq \nu(t) > a$ for all $n$ positive integer, and $\nu(z) = \nu(t^{-1})$ for each $z \in F \setminus \{0\}$ so that $\nu(z) < a$, that is, if $z = ut^n$ with $u$ a unit of $\nu_a$ and $n$ a negative integer, then $\nu(z) = \nu(t^{-1})$.*

(*ix*) *If* $u$ *is a unit of* $\nu_a$ *and* $z \in F \setminus \{0\}$ *such that* $\nu(z) < a$*, then* $\nu(uz) = \nu(u) \wedge \nu(z) = \nu(z)$*;*

(*x*) *if there exists* $w \in F$ *such that* $\nu(w) = 0$*, then* $\nu^* = \nu_a$*.*

**Proof.** $(i)$: Let $z \in F \setminus \{0\}$ be so that $\nu(z) > a$. Then, $\nu(0) \geq \nu(z) > a$.
$(ii)$: By definition of $\nu$, it follows immediately that $K \subsetneq \nu_a \subsetneq F$.
$(iii)$: Let $z \in F \setminus \{0\}$ be such that $\nu(z) < a$, then $\nu(z^{-1}) > a$. Hence, $\nu(z) \geq a$ or $\nu(z^{-1}) \geq a$, for all $z \in F \setminus \{0\}$.
$(iv)$: Since $\nu_a$ is a subring of $F$, the statement follows from $(ii)$ and $(iii)$.
$(v)$ The proof will be by contradiction. We assume that there exists $z \in \tilde{K}$ such that $\nu(z) < a$, or equivalently $\nu(z^{-1}) > a$, since $\nu$ is a valuation. So, we have that $z^{-1}$ is algebraic over $K$, with $z^{-1} \notin K$. Let $c_1, \ldots, c_n \in K$ be such that $c_n(z^{-1})^n + \cdots + c_1 z^{-1} + 1 = 0$. Then, $z^{-1}(c_n(z^{-1})^{n-1} + \cdots + c_1) = -1$, or equivalently, $z = -(c_n(z^{-1})^{n-1} + \cdots + c_1)$. Then

$$\begin{aligned}
\nu(z) &= \nu(-(c_n(z^{-1})^{n-1} + \cdots + c_1)) = \nu(c_n(z^{-1})^{n-1} + \cdots + c_1) \\
&\geq \nu(c_n(z^{-1})^{n-1}) \wedge \cdots \wedge \nu(c_2 z^{-1}) \wedge \nu(c_1) \\
&\geq (\nu(c_n) \wedge \nu(z^{-1})^{n-1}) \wedge \cdots \wedge (\nu(c_2) \wedge \nu(z^{-1})) \wedge \nu(c_1) \\
&\geq (a \wedge \nu(z^{-1})^{n-1})) \wedge \cdots \wedge (a \wedge \nu(z^{-1})) \wedge a \\
&\geq a \\
&> \nu(z),
\end{aligned}$$

by this is absurd. Hence, $\nu(z) \geq a$.
On the other hand, if $z \in \tilde{K}$, with $z \neq 0$, we have that $\nu(z) \geq a$. But, if $\nu(z) > a$, it would have to $\nu(z^{-1}) < a$, which is absurd, since $\nu(z^{-1}) \geq a$, with $z^{-1} \in K$. Therefore, $\nu(z) = a$.
$(vi)$: We assume $\nu(u) = a$. If $u$ would not be unity of $\nu_a$, we will have that $u^{-1} \notin \nu_a$, that is $\nu(u^{-1}) < a$, with which $\nu(u) > a$, but this is absurd. Then, $u$ is a unity of $\nu_a$. Reciprocally, if $u$ is a unity of $\nu_a$, we will have that $\nu(u) \geq a$ and $\nu(u^{-1}) \geq a$. If $\nu(u) > a$, then $\nu(u^{-1}) < a$, contradiction. So that, necessarily $\nu(u) = a$.
$(vii)$: Since $\nu(zw) \geq a$, we have that if $\nu(zw) = a$, then $zw$ would be unity of $\nu_a$, that is, there exists $u \in \nu_a$ so that $z(wu) = 1$ where $wu \in \nu_a$. This would imply that $z$ would be a unity of $\nu_a$ and $\nu(z) = a$, which is a contradiction. Therefore, $\nu(zw) > a$.
$(viii)$: In accordance with $(iv)$, it is clear that $\nu(t) > a$, for which $\nu(t^{-1}) < a$. If $z \in F \setminus \{0\}$ satisfies that $\nu(z) < a$, then $z \notin \nu_a$, with which $z = ut^{-n}$ for some unity $u$ of $\nu_a$ and $n \in \mathbb{N}$. Then, as $\nu(t^{-n}) < a$, we have

$$\nu(z) = \nu(ut^{-n}) \geq \nu(u) \wedge \nu(t^{-n}) = a \wedge \nu(t^{-n}) = \nu(t^{-n}) = \nu((t^{-1})^n) \geq \nu(t^{-1}),$$

that is $\nu(z) \geq \nu(t^{-1})$. On the other hand,

$$\nu(t^{-1}) = \nu(uu^{-1}t^{n-1}t^{-(n-1)}t^{-1}) \geq \nu(u^{-1}) \wedge \nu(t^{n-1}) \wedge \nu(ut^{-n}) \geq a \wedge \nu(z) = \nu(z),$$

that is $\nu(t^{-1}) \geq \nu(z)$. Therefore, $\nu(z) = \nu(t^{-1})$.
$(ix)$: It holds immediately because of $(viii)$.
$(x)$: If $\nu(w) = 0$ for some $w \in F$ then, necessarily, $w \neq 0$ and $\nu(w) = \nu(t^{-1}) = 0$. Then, for each $z \in F$ such that $\nu(z) > 0$, it must have to $\nu(z) \geq a$, that is, $z \in \nu_a$. Therefore, $\nu^* = \nu_a$.

**Proposition 2.2.** *Let $\nu$ be an $L$-subring of valuation of a function field $F/K$, with $a$ its divisor element. Assume that $\nu$ has the property that, for all $x \in F$ nonzero, the relation $\nu(x) = a$ implies that $\nu(xy) \geq \nu(y)$ for all $y \in F$. Then, for all $x \in F$ nonzero, $\nu(x) = a$ if and only if $\nu(xy) = \nu(y)$ for all $y \in F$.*
**Proof.** Let $x \in F$ be arbitrary. We assume that $\nu(x) = a$ and let $y \in F$. We have that $\nu(x) \geq \nu(xy)$. On the other hand, as $\nu(x^{-1}) = a$, it holds that $\nu(y) = \nu(x^{-1}(xy)) \geq \nu(xy)$. Hence, $\nu(xy) = \nu(y)$. Reciprocally, we assume that $\nu(xy) = \nu(y)$ for all $y \in F$. If $\nu(x) > a$, then $a = \nu(1) = \nu(1 \cdot x) = \nu(x) > a$, which is absurd. Similarly, it must not happen that $\nu(x) < a$. Therefore, $\nu(x) = a$.
Let $+\infty$ be an object such that $+\infty \notin \mathbb{Z}$, $+\infty > n$ and $(+\infty)+n = n+(+\infty) = +\infty$ for all $n \in \mathbb{Z}$.

**Theorem 2.1.** *Let $F/K$ a function field, $P$ a place of $F$ and $\upsilon_P$ the valuation associated with $P$. Then, for all $a \in L$ with $0 < a < 1$, there exists a function $\varphi_a : \mathbb{Z} \cup \{+\infty\} \to L$ such that $\varphi_a \circ \upsilon_P$ is an $L$-subring of valuation of $F$ with divisor element $a$.*

*Reciprocally, let $\nu$ be an $L$-subring of valuation of the function field $F/K$ with divisor element $a$. Then, there exists a function $\varphi : \mathbb{Z} \cup \{+\infty\} \to L$ such that $\varphi \circ \upsilon_P$ is an $L$-subring of valuation of $F$ and $\varphi \circ \upsilon_P \subseteq \nu$ where $P$ is the place associated with $\nu_a$.*
**Proof.** We will prove the first part of the theorem. Let $a \in L$ be such that $0 < a < 1$. Then, we build a function $\varphi_a : \mathbb{Z} \cup \{+\infty\} \to L$ with the following properties:

$(a)$ For all $m, n \in \mathbb{Z} \cup \{+\infty\}$, the relation $m \leq n$ implies that $\varphi_a(m) \leq \varphi_a(n)$;

$(b)$ $\varphi_a(m+n) \geq \varphi_a(m) \wedge \varphi_a(n)$ for all $m, n \in \mathbb{Z} \cup \{+\infty\}$;

$(c)$ $\varphi_a(0) = a$;

$(d)$ $\varphi_a(1) > a$;

(*e*) for all $n \in \mathbb{Z} \setminus \{0\}$, $\varphi_a(n) > a$ if and only if $\varphi_a(-n) < a$.

Then, the properties (*a*) and (*b*) of $\varphi_a$ establish that $\varphi \circ \upsilon_P$ is an $L$-subring of $F$; while the properties (*a*), (*c*), (*d*) and (*e*) justify that $\varphi \circ \upsilon_P$ is an $L$-subring of valuation of $F$, according to the Definition 2.1.

Reciprocally, let $t$ be a prime element of $\nu_a$. For each $n \in \mathbb{Z}$, $n \geq 1$, we denote by $A_n$ to the set $\{\nu(ut^k) \mid u \text{ is a unity of } \nu_a \text{ and } k \geq n\}$. Then, we define the function $\varphi : \mathbb{Z} \cup \{+\infty\} \to L$ as follows: $\varphi(+\infty) = \nu(0)$, and for all $n \in \mathbb{Z}$,

$$\varphi(n) := \begin{cases} \bigwedge\limits_{x \in A_n} x & \text{if} \quad n \geq 1 \\ a & \text{if} \quad n = 0 \\ \nu(t^{-1}) & \text{if} \quad n < 0 \end{cases}$$

Note that, for all $n \in \mathbb{Z}$ with $n \geq 1$, $A_n \supseteq A_{n+1}$, with which $\varphi(n) \leq \varphi(n+1)$. This implies that $\varphi(m+n) \geq \varphi(m) \wedge \varphi(n)$ for all $m, n \in \mathbb{Z}$ with $m, n \geq 1$. Consequently, we have that $\varphi$ satisfies the conditions (*a*)-(*e*) and, hence, $\varphi \circ \upsilon_P$ is an $L$-subring of valuation of $F$, where $P$ is the place associated with $\nu_a$. Finally, the fact that $\varphi \circ \upsilon_P \subseteq \nu$ follows from the definition of $\varphi$.

**Observation 2.1.** In the reciprocal of Theorem 2.1, if the $L$-subring $\nu$ of valuation of the function field $F/K$ with divisor element $a$ satisfies that $\nu(t^n) \leq \nu(t^{n+1})$ and $\nu(ut^n) \geq \nu(t^n)$ for all $u$ unit of $\nu_a$ and for all $n$ positive integer, then the function $\varphi : \mathbb{Z} \cup \{+\infty\} \to L$ in the proof of the theorem is given by: $\varphi(+\infty) = \nu(0)$ and for each $n \in \mathbb{Z}$

$$\varphi(n) := \begin{cases} \nu(t^n) & \text{if} \quad n \geq 1 \\ a & \text{if} \quad n = 0 \\ \nu(t^{-1}) & \text{if} \quad n < 0. \end{cases}$$

In particular, if $\nu(ut^n) = \nu(t^n)$ for all $u$ unit of $\nu_a$ and for all $n$ positive integer, then $\varphi \circ \upsilon_P = \nu$.

**Proposition 2.3.** *Let $\nu$ be an $L$-subring of valuation of a function field $F/K$ with divisor element $a$, $\psi$ an $L$-subring of $F$ and $b \in L$ such that $0 < b < 1$. Then, $\psi$ is an $L$-subring of valuation of $F$ with divisor element $b$ such that $\psi_b = \nu_a$ if and only if $\psi$ satisfies the following properties*

(*i*) *$b$ and $\psi(z)$ are comparable for all $z \in F$;*

*(ii) for all $z \in F$, $\psi(z) = b$ if and only if $\nu(z) = a$;*

*(iii) for all $z \in F$, $\psi(z) > a$ if and only if $\nu(z) > b$.*

**Proof.** Suppose that $\psi$ is an $L$-subring of valuation of the function field $F/K$ with divisor element $b$ such that $\psi_b = \nu_a$. It is clear that the condition $(i)$ is true. We will prove $(ii)$. Let $z \in F$. Then, $\psi(z) = b$ if and only if $b$ is a unit of $\psi_b = \nu_a$, if and only if $\nu(z) = a$. Now $(iii)$ is immediate from $(ii)$.
Reciprocally, it is clear $(ii)$ y $(iii)$ imply that $\psi_b = \nu_a$; while the properties $(i)$-$(iv)$ from the Definition 2.1 are obtained from the hypotheses.

**Corollary 2.1.** *Let $\nu$ be an $L$-subring of valuation of a function field $F/K$ with divisor element $a$, $\psi$ an $L$-subring of $F$ such that $\nu \subseteq \psi$, $a$ and $\psi(z)$ are comparable for all $z \in F$ and, for all $z \in F$, $\psi(z) = a$ if and only if $\nu(z) = a$. Then, $\psi$ is an $L$-subring of valuation of $F$ with divisor element $a$.*
**Proof.** Since $\nu \subseteq \psi$, we have that $\nu_a \subseteq \psi_a$ with $\psi_a$ a proper subring of $F$; but, $\nu_a$ is a maximal proper subring of $F$, then necessarily $\nu_a = \psi_a$. Hence, it is enough to prove $(iii)$ of the Proposition 2.3 with $b = a$. Then, let $z \in F$. If $\psi(z) > a$, then $\nu(z) \geq a$ since $z \in \psi_a = \nu_a$. For hypothesis, necessarily $\nu(z) > a$. Reciprocally, if $\nu(z) > a$, then $\psi(z) \geq \nu(z) > a$. This completes the proof.

## 3. An Approximation Theorem

**Proposition 3.1.** *Let $\nu$ be an $L$-subring of a function field $F/K$. For each $x, y \in F$, it is true that*

$$\nu(x+y) \wedge \nu(x) = \nu(x) \wedge \nu(y) = \nu(x+y) \wedge \nu(y).$$

*In particular, if $\{\nu(x), \nu(y), \nu(x+y)\}$ is a chain in $L$ with $\nu(x) \neq \nu(y)$, then*

$$\nu(x+y) = \nu(x) \wedge \nu(y).$$

**Proof.** Since $\nu(x) \geq \nu(x) \wedge \nu(y)$, $\nu(y) \geq \nu(x) \wedge \nu(y)$ and $\nu(x+y) \geq \nu(x) \wedge \nu(y)$, by definition of the concept of minimum, we obtain

$$\nu(x+y) \wedge \nu(x) \geq \nu(x) \wedge \nu(y) \quad \text{and} \quad \nu(x+y) \wedge \nu(y) \geq \nu(x) \wedge \nu(y).$$

On the other hand, as $\nu(x) = \nu((x+y) - y) \geq \nu(x+y) \wedge \nu(y)$, we have that $\nu(x) \wedge \nu(y) \geq (\nu(x+y) \wedge \nu(y)) \wedge \nu(y) = \nu(x+y) \wedge \nu(y)$; similarly, $\nu(x) \wedge \nu(y) \geq \nu(x+y) \wedge \nu(x)$. Hence,

$$\nu(x+y) \wedge \nu(x) = \nu(x) \wedge \nu(y) = \nu(x+y) \wedge \nu(y).$$

In particular, if $\nu(x) < \nu(y)$, as $\nu(x+y) \geq \nu(x) \wedge \nu(y) = \nu(x)$ it holds that $\nu(x) = \nu(x+y) \wedge \nu(x) = \nu(x+y) \wedge \nu(y) = \nu(x+y)$, because $\nu(x+y) \wedge \nu(y) \neq \nu(y)$.

**Corollary 3.1.** *Let $\nu$ be an $L$-subring of the function field $F/K$, $u$ a unit of $\nu_a$ and $z \in F$ nonzero such that $\nu(z) \neq a$ and $\{\nu(z), \nu(z+u)\}$ is a chain in $L$. Then $\nu(z+u) = \nu(z) \wedge a$.*

**Proof.** It follows immediately from the Proposition 3.1.

**Definition 3.1.** *Two $L$-subrings of valuation $\nu_1$ and $\nu_2$ of a function field $F/K$, with divisor elements $a_1$ and $a_2$ respectively, are* ***L-equivalent*** *if $\nu_{a_1} = \nu_{a_2}$.*

**Theorem 3.1. (Approximation Theorem for $L$-Subrings of Valuation)** *Let $\nu_1, \ldots, \nu_n$ be $L$-subrings of valuation of a function field $F/K$, $n \geq 2$, with divisor elements $a_1, \ldots, a_n$ respectively, which satisfy the following properties: For all $i, j = 1, \ldots, n$,*

*(i)* *$\nu_i$ and $\nu_j$ are not $L$-equivalent if $i \neq j$;*

*(ii)* *$\nu_i(F)$ is a chain in $L$;*

*(iii)* *if $z_1, z_2 \in F$ are such that $\nu_i(z_1) \geq \nu_i(z_2)$, then $\nu_i(z_1 z) \geq \nu_i(z_2 z)$ for each $z \in F$;*

*(iv)* *if $z_1, z_2 \in F$ with $\nu_i(z_1) > a_i$ and $\nu_i(z_2) > a_i$, then there exists $m \in \mathbb{N}$ such that $\nu_i(z_1^m) > \nu_i(z_2)$;*

*(v)* *if $z_1, z_2 \in F$ are nonzero such that $\nu_i(z_1) < a_i$ and $\nu_i(z_2) < a_i$, with $\nu_i(z_1^{-1}) \neq \nu_i(z_2^{-1})$, then $\nu_i(z_1 + z_2) < a_i$.*

*Then, for each $x_1, \ldots, x_n \in F$ and for each $r_i \in \nu_i(F)$, with $r_i \neq \nu_i(0)$ and $i = 1, \ldots, n$, there exists $x \in F$ such that*

$$\nu_i(x - x_i) = r_i, \quad \textit{for each} \quad i = 1, \ldots, n.$$

**Proof.** We will prove the theorem through several steps.

**Affirmation I:** *There exists $u \in F$ such that $\nu_1(u) > a_1$ and $\nu_i(u) < a_i$ for all $i = 2, \ldots, n$.*

We will proceed by induction on $n$. For $n = 2$, we have that $\nu_{a_1} \nsubseteq \nu_{a_2}$ and $\nu_{a_2} \nsubseteq \nu_{a_1}$. Let $y_1$ and $y_2$ be elements of $F$ nonzero such that $\nu_1(y_1) \geq a_1$, $\nu_2(y_1^{-1}) > a_2$, $\nu_2(y_2) \geq a_2$ and $\nu_1(y_2^{-1}) > a_1$. Let $u := y_1 y_2^{-1}$. Then, we have that $\nu_1(u) > a_1$ and $\nu_2(u^{-1}) > a_2$, that is, $\nu_1(u) > a_1$ and $\nu_2(u) < a_2$. Hence, we assume that the statement holds for $n$. There exists $y \in F$ such that $\nu_1(y) > a_1$ and $\nu_i(y) < a_i$ for all $i = 2, \dots, n$. If $\nu_{n+1}(y) < a_{n+1}$ then we would have finished, whence we assume that $\nu_{n+1}(y) \geq a_{n+1}$. Let $z \in F$ be such that $\nu_1(z) > a_1$ and $\nu_{n+1}(z) < a_{n+1}$. Observe that if $\nu_i(z) < a_i$ for all $i = 2, \dots, n$, then again the proof would be finished. Hence, changing the numbering if necessary, we can assume that there exists $2 \leq k < n+1$ such that $\nu_i(z) \geq a_i$ for all $2 \leq i \leq k$, and $\nu_j(z) < a_j$ for all $k < j \leq n+1$. By hypothesis, for all $i = k+1, \dots, n$, there exists $l_i$ a positive integer such that $\nu_i(z^{-l_i}) > \nu_i(y^{-1})$. Taking $l = \max\{l_i \mid k+1 \leq i \leq n\}$, we have that $\nu_i(z^{-l}) > \nu_i(y^{-1})$ for all $i = k+1, \dots, n$. By $(v)$, we have that $\nu_i(y + z^l) < a_i$ for all $i = k+1, \dots, n$. Since $\nu_1(z^l) \geq \nu_1(z) > a_1$, $\nu_i(z^l) \geq \nu_1(z) \geq a_i$ for all $i = 2, \dots, k$, and $\nu_{n+1}(z^{-l}) \geq \nu_{n+1}(z^{-1}) > a_{n+1}$, we obtain that $\nu_1(y+z^l) > a_1$ and $\nu_i(y + z^l) < a_i$ for all $i = 2, \dots, n+1$. So, it is enough to choose $u = y + z^l$. This complete the inductive process.

**Affirmation II:** *There exists $w \in F$ such that $\nu_1(w-1) > r_1$ and $\nu_i(w) > r_i$ for all $i = 2, \dots, n$.*

Firstly, we note that the hypothesis $(iii)$ implies that $\nu_i(zv) = \nu_i(z)$ for each $z \in F$ and for all $v$ unit in $(\nu_i)_{a_i}$, for $i = 1, \dots, n$. By Affirmation I, let $u \in F$ be such that $\nu_1(u) > a_1$ and $\nu_i(u^{-1}) > a_i$ for all $i = 2, \dots, n$. Then, by hypothesis $(iv)$, there exist $m_1, \dots, m_n \in \mathbb{N}$ such that $\nu_1(u^{m_1}) > a_1 \vee r_1$ and $\nu_i(u^{-m_i}) > a_i \vee r_i$ for all $i = 2, \dots, n$. Let $m = m_1 \cdots m_n$, then $\nu_1(u^m) \geq \nu_1(u^{m_1}) > a_1 \vee r_1$ and $\nu_i(u^{-m}) \geq \nu_i(u^{-m_i}) > a_i \vee r_i$ for all $i = 2, \dots, n$. We write $w = 1/(1+u^m)$. Then, $\nu_1(w-1) = \nu_1(-u^m/(1+u^m)) = \nu_1(u^m) > r_1$, since $-1/(1+u^m)$ is a unit in $(\nu_1)_{a_1}$ and, on the other hand, by $(iii)$

$$\nu_i(w) = \nu_i\left(\frac{1}{1+u^m}\right) = \nu_i\left(\frac{1}{u^m} - \frac{1}{u^m(1+u^m)}\right) \geq \nu_i\left(\frac{1}{u^m}\right) > r_i,$$

where $\nu_i(w) = \nu_i(1/1+u^m)) > a_i = \nu_i(1)$.

**Affirmation III:** *For each $y_1, \dots, y_n \in F$, there exists $z \in F$ such that $\nu_i(z - y_i) > r_i$, for all $i = 1, \dots, n$.*

We have that if $y_i = 0$ for all $i = 1, \dots, n$, then it is enough to choose $z = 0$, so we will assume that not all elements $y_i$, $i \in \{1, \dots, n\}$, are zero.

By Affirmation II, para cada $i = 1, \dots, n$, we choose $w_i \in F$ such that

$$\nu_i(w_i - 1) > \begin{cases} r_i & \text{if} \quad y_i = 0 \\ \nu_i\left(\dfrac{z_i}{y_i}\right) & \text{if} \quad y_i \neq 0 \end{cases} \quad \text{and} \quad \nu_j(w_i) > \begin{cases} r_j & \text{if} \quad y_j = 0 \\ \nu_j\left(\dfrac{z_j}{y_j}\right) & \text{if} \quad y_j \neq 0\ , \end{cases}$$

where $z_j$ is an element in $F$ such that $\nu_j(z_j) > r_j$, by $(iv)$, for all $j = 1, \dots, n$.

Then, defining $z := \sum_{l=1}^{n} y_l w_l$, we have, by applying hypothesis $(iii)$ where necessary, that $\nu_i(z - y_i) > r_i$ for each $i = 1, \ldots, n$.

Finally, let us conclude the proof of the theorem. By Affirmation III, we take elements $z, z_1, \ldots, z_n$ in $F$ such that $\nu_i(z - x_i) > r_i$ and $\nu_i(z_i) = r_i$ for all $i = 1, \ldots, n$; furthermore, we choose an element $z' \in F$ so that $\nu_i(z' - z_i) > r_i$ for each $i = 1, \ldots, n$, by Affirmation III. Thus,

$$\nu_i(z') = \nu_i((z' - z_i) + z_i) = \nu_i(z' - z_i) \wedge \nu_i(z_i) = \nu_i(z_i) = r_i,$$

for all $i = 1, \ldots, n$. We define $x := z + z'$. Then

$$\nu_i(x - x_i) = \nu_i((z - x_i) + z') = \nu_i(z - x_i) \wedge \nu_i(z') = \nu_i(z') = r_i,$$

for all $i = 1, \ldots, n$. This complete the proof.

## 4. Valuations of function fields onto a lattice

**Definition 4.1.** *Let $F/K$ be a function field, $L$ a lattice and $\mu$ an $L$-subset of $F$. We say that $\mu$ is an **L-valuation of the function field** $F/K$ if $\mu$ satisfies the following properties:*

- $(i)$ *For each $x \in F$, $x \neq 0$, $0 < \mu(x) < \mu(0) \leq 1$;*
- $(ii)$ *there exists $a \in L$, with $0 < a < \mu(0)$, such that $\mu(x) \geq a$ or $\mu(x^{-1}) \geq a$, for each $x \in F$ nonzero;*
- $(iii)$ *$\mu(x + y) \geq \mu(x) \wedge \mu(y)$, for all $x, y \in F$;*
- $(iv)$ *for all $x, y \in F$,*
  - $(a)$ *$\mu(xy) \geq \mu(x) \vee \mu(y)$ if $\mu(x) \geq a$ and $\mu(y) \geq a$*
  - $(b)$ *$\mu(x) < \mu(xy) < \mu(y)$ if $\mu(x) < a$ and $\mu(y) > a$*
  - $(c)$ *$\mu(xy) \leq \mu(x) \wedge \mu(y)$ if $\mu(x) \leq a$ and $\mu(y) \leq a$;*
- $(v)$ *$\mu(x) = a$ for all $x \in K$, with $x \neq 0$;*
- $(vi)$ *there exists $z \in F$ nonzero so that $\mu(z) > a$.*

*We will call the element $a$ of $L$ again **divisor element** of $\mu$.*

**Proposition 4.1.** *Under the conditions of the Definition 4.1, for all nonzero $x, y, z \in F$,* (*i*) $\mu(x) \geq a$ *and* $\mu(x^{-1}) \geq a$ *implies that* $\mu(x) = \mu(x^{-1}) = a$;

(*ii*) $\mu(x) > a$ *if and only if* $\mu(x^{-1}) < a$;

(*iii*) $\mu(x) = a$ *if and only if* $\mu(x^{-1}) = a$;

(*iv*) *if* $\mu(x) = a = \mu(y)$, *then* $\mu(xy) = a$;

(*v*) $\mu(x) = a$ *if and only if* $\mu(xy) = \mu(y)$. *In particular,* $\mu(-y) = \mu(y)$;

(*vi*) $\mu(x/y) = a$ *if and only if* $\mu(x) = \mu(y)$;

(*vii*) $\mu(x/y) > a$ *if and only if* $\mu(y) < \mu(x)$;

(*viii*) $\mu(x/y) < a$ *if and only if* $\mu(x) < \mu(y)$;

(*ix*) $\mu(x) > \mu(y)$ *if and only if* $\mu(xz) > \mu(yz)$;

(*x*) *the image* $\mu(F)$ *is a chain in the lattice* $L$;

(*xi*) $\mu(x) < \mu(y)$ *if and only if* $\mu(y^{-1}) < \mu(x^{-1})$;

(*xii*) *The set* $\mu_a = \{w \in F \mid \mu(w) \geq a\}$ *is a subring of valuation of* $F/K$;

(*xiii*) $x$ *is a unity of* $\mu_a$ *if and only if* $\mu(x) = a$;

(*xiv*) *if* $\mu(x) > a$ (*respectively* $\mu(x) < a$; $\mu(x) = a$), *then* $\mu(x^n) < \mu(x^{n+1})$ (*respectively* $\mu(x^n) > \mu(x^{n+1})$; $\mu(x^n) = a$), *for all* $n \geq 1$. *In particular, the image* $\mu(F)$ *of* $\mu$ *is an infinite set.*

(*xv*) *If* $\tilde{K}$ *is the constant field of the function field* $F/K$, *then* $\mu(x) = a$, *when* $x \in \tilde{K}$ $(x \neq 0)$.

**Proof.** (*i*): We have that $a = \mu(1) = \mu(xx^{-1}) \geq \mu(x) \vee \mu(x^{-1}) \geq \mu(x), \mu(x^{-1}) \geq a$, that is, $\mu(x) = \mu(x^{-1}) = a$.

(*ii*): By (*i*), if $\mu(x) > a$, then it cannot happen that $\mu(x^{-1}) \geq a$, for which $\mu(x^{-1}) < a$.

(*iii*): It follows from (*ii*).

(*iv*): If $\mu(x) = a = \mu(y)$, then $\mu(xy) \geq \mu(x) \vee \mu(y) = a$ and $\mu(xy) \leq \mu(x) \wedge \mu(y) = a$, for which $\mu(xy) = a$.

(*v*): We assume that $\mu(x) = a$. If $\mu(y) > a$, we will have that $\mu(xy) \geq \mu(x) \vee \mu(y) = \mu(y) > a$ and, on the other hand, $\mu(y) = \mu(x^{-1}(xy)) \geq \mu(x^{-1}) \vee \mu(xy) = \mu(xy)$,

because $\mu(x^{-1}) = a$ and $\mu(xy) > a$. Hence, $\mu(xy) = \mu(y)$. Similarly, equality is proven in the case that $\mu(y) < a$. Of course, if $\mu(y) = a$, then the affirmation follows from $(iv)$.
Reciprocally, we assume that $\mu(xy) = \mu(y)$. If $\mu(x) > a$, then it holds that $\mu(y) \geq a$ or $\mu(y) < a$. In the first case, we have that $\mu(y) = \mu(xy) \geq \mu(x) \vee \mu(y)$, that is, $\mu(y) = \mu(x) \vee \mu(y)$, for which $a < \mu(x) \leq \mu(y)$; thus, $\mu(xy) = \mu(y) = \mu(x^{-1}(xy)) < \mu(xy)$, but this is absurd. In the second case, we will have the inequality $\mu(y) < \mu(xy) < \mu(x)$, the which also is absurd. However, if $\mu(x) < a$, similarly, it may happen that $\mu(y) > a$ or $\mu(y) \leq a$. In the first case, it will have the inequality $\mu(x) < \mu(xy) < \mu(y)$, the which is absurd. If $\mu(y) \leq a$, we will have that $\mu(y) = \mu(xy) \leq \mu(x) \wedge \mu(y)$, for which $\mu(y) \leq \mu(x) < a$; thus, $\mu(xy) < \mu(x^{-1}(xy)) = \mu(y) = \mu(xy)$, since $\mu(x^{-1}) > a$ and $\mu(xy) < a$, but this is absurd. Therefore, necessarily, $\mu(x) = a$.

$(vi)$: If $\mu(x/y) = a$, we have that $\mu(x) = \mu((x/y)y) = \mu(y)$. Reciprocally, if $\mu(x) = \mu(y)$, we have that $\mu(y) = \mu(x) = \mu((x/y)y)$, this implies that $\mu(x/y) = a$.

$(vii)$: We assume that $\mu(x/y) > a$. By $(vi)$, we obtain that $\mu(x) \neq \mu(y)$. We have the following cases:

Case $\mu(x) < a$: it cannot happen that $\mu(y) \geq a$. In fact, if $\mu(y) > a$, then $\mu(x/y) \leq \mu(x) \wedge \mu(y^{-1}) \leq a$, the which is absurd. If $\mu(y) = a$, then $\mu(y^{-1}) = a$ and $\mu(x/y) = \mu(x) < a$, the which also is absurd. Thus, necessarily, $\mu(y) < a$ and, with this, $\mu(y) < \mu((x/y)y) < \mu(x/y)$, that is, $\mu(y) < \mu(x)$.

Case $\mu(x) = a$: Here, it cannot happen that $\mu(y) \geq a$. Since, if $\mu(y) > a$, we will have that $a = \mu(x) = \mu((x/y)y) \geq \mu(x/y) \vee \mu(y) \geq \mu(y) > a$, the which is absurd. If $\mu(y) = a$, it will have that $\mu(x/y) = \mu(x) = a$, neither can it be. Thus, necessarily, $\mu(y) < a$, that is, $\mu(y) < a = \mu(x)$.

Case $\mu(x) > a$: Regardless of the value that $\mu(y)$ takes, we will see that $\mu(y) < \mu(x)$. If $\mu(y) > a$, then $\mu(x) = \mu((x/y)y) \geq \mu(x/y) \vee \mu(y) \geq \mu(y)$, with $\mu(x) \neq \mu(y)$, that is, $\mu(y) < \mu(x)$. If $\mu(y) = a$, then $\mu(y) = a < \mu(x/y) = \mu(x)$. Finally, if $\mu(y) < a$, then $\mu(y) < a < \mu(x)$.

In summary, if $\mu(x/y) > a$, then $\mu(y) < \mu(x)$. Reciprocally, we assume that $\mu(y) < \mu(x)$. If we had to $\mu(x/y) < a$, then $\mu(y/x) > a$ and, on the other hand, we must have that $\mu(x) < \mu(y)$, but this is absurd. If $\mu(x/y) = a$, then $\mu(x) = \mu(y)$, the which cannot happen either. Necessarily, $\mu(x/y) > a$.

$(viii)$: It is an immediate consequence of $(vi)$ and $(vii)$.

$(ix)$: $\mu(x) > \mu(y) \Leftrightarrow \ \mu\left(\dfrac{xz}{yz}\right) = \mu\left(\dfrac{x}{y}\right) > a \Leftrightarrow \ \mu(xz) > \mu(yz)$.

$(x)$: It is immediate.

$(xi)$: $\mu(x) < \mu(y) \Leftrightarrow \mu\left(\frac{y}{x}\right) > a \Leftrightarrow \mu\left(\frac{y^{-1}}{x^{-1}}\right) < a \Leftrightarrow \mu(y^{-1}) < \mu(x^{-1})$.

$(xii)$: It is immediate.

$(xiii)$: It is immediate of $(i)$ and $(iii)$.

$(xiv)$: Assume that $\mu(x) > a$, then $\mu(x^{n+1}/x^n) = \mu(x) > a$ for all $n \in \mathbb{N}$, that is, $\mu(x^{n+1}) > \mu(x^n)$ for all $n \in \mathbb{N}$. Similarly, we obtain the results for the cases $\mu(x) = a$ or $\mu(x) < a$ when applying $(vi)$ and $(viii)$ respectively.

$(xv)$: Suppose that $x \in \tilde{K}$ $(x \neq 0)$. We assume that $\mu(x) > a$. We take $f(x) = irr(x, K) = a_m X^m + a_{m-1}X^{m-1} + \cdots + a_1 X + a_0$, with $a_m = 1$ and $a_0 \neq 0$. Then, $a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 = 0$, where $\mu(a_i x^i) = \mu(x^i) > \mu(x^{i-1}) = \mu(a_{i-1}x^{i-1})$ for each $i = 1, \ldots, m$. Hence, $\mu(0) = \mu(a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0) = \mu(a_m x^m) \wedge \cdots \wedge \mu(a_0) = \mu(a_0) < \mu(0)$, but this is absurd. Similarly, we can prove that the relationship $\mu(x) < a$ give a contradiction. Therefore, necessarily, $\mu(x) = a$.

**Corollary 4.1.** *Let $\mu$ be an $L$-valuation of a function field $F/K$ with divisor element $a$. Then, there exists an injective homomorphism of lattices $\psi : \mathbb{Z} \longrightarrow L$ such that $0 < \psi(m) < 1$ for each $m \in \mathbb{Z}$.*

**Proof.** Since $\mu_a$ is a subring of valuation of $F$, let $t \in \mu_a$ be a prime element of the place $P$ associated with $\mu_a$. Then, each element $x \in F$ nonzero is uniquely expressed as $x = ut^n$ for some $u$ unit of $\mu_a$ and for some $n \in \mathbb{Z}$, where $\mu(x) = \mu(t^n)$. Thus, an injective homomorphism of lattices $\psi : \mathbb{Z} \longrightarrow L$ required is given by $\psi(n) := \mu(t^n)$ for all $n \in \mathbb{Z}$.

**Proposition 4.2.** *Let $F/K$ be a function field, $L$ a lattice, $\mu$ an $L$-valuation of $F/K$ with divisor element $a$ and $x, y, x_1, \ldots, x_n \in F$.*

*(i) If $\mu(x_i) \neq \mu(x_j)$ for all $i \neq j$, then $\mu(x_1 + \cdots + x_n) = \mu(x_1) \wedge \cdots \wedge \mu(x_n)$;*

*(ii) if $x_1 + \cdots + x_n = 0$, then there exist $i \neq j$ such that $\mu(x_i) = \mu(x_j)$;*

*(iii) if $\mu(x) < \mu(y) \leq a$, then $\mu(x + y) < a$.*

**Proof.** $(i)$: By induction on $n$. If $n = 1$, then the affirmation is clear. We will prove it to $n = 2$. We can assume that $\mu(x_1) < \mu(x_2)$. Then, $\mu(x_1) = \mu((x_1 + x_2) - x_2) \geq \mu(x_1 + x_2) \wedge \mu(x_2) = \mu(x_1 + x_2) \geq \mu(x_1) \wedge \mu(x_2) = \mu(x_1)$, that is, $\mu(x_1 + x_2) = \mu(x_1) = \mu(x_1) \wedge \mu(x_2)$. Now, we assume the affirmation hold to $n$, and we will prove it for $n+1$. We note that $\mu(x_1) \wedge \cdots \wedge \mu(x_n) \neq \mu(x_{n+1})$, with which it has that $\mu(x_1+\cdots+x_{n+1}) = \mu((x_1+\cdots+x_n)+x_{n+1}) = \mu(x_1+\cdots+x_n)\wedge\mu(x_{n+1}) = (\mu(x_1) \wedge \cdots \wedge \mu(x_n)) \wedge \mu(x_{n+1}) = \mu(x_1) \wedge \cdots \wedge \mu(x_{n+1})$. This complete the proof.

$(ii)$: It follows immediately from $(i)$.

$(iii)$ It is immediate.

Let $\mu$ and $\mu_1$ be $L$-valuations of the function field $F/K$ with $a$ and $a_1$ divisor elements, respectively. Again, we say that $\mu$ and $\mu_1$ are **equivalent** if $\mu_a = (\mu_1)_{a_1}$. On the other hand, we say that $\mu$ satisfies the **Archimedean property** if for each $x, y \in F$, with $\mu(x) > a$ and $\mu(y) > a$, there exists $m \in \mathbb{N}$ so that $\mu(x^m) > \mu(y)$.

**Theorem 4.1.** *Let $L$ and $L_1$ be two lattices, $\mu$ an $L$-valuation and $\mu_1$ a $L_1$-valuation both of a function field $F/K$ with divisor elements $a$ and $b$ respectively. Then, $\mu_a = (\mu_1)_b$ if and only if there exists an isomorphism of lattices $\psi : \mu(F) \longrightarrow \mu_1(F)$ such that maps $a$ onto $b$ and $\psi \circ \mu = \mu_1$.*

**Proof.** We note that $\mu(F)$ and $\mu_1(F)$ are sublattices of $L$ and $L_1$ respectively. Assume that $\mu_a = (\mu_1)_b$, that is, it is true that, for each $x \in F$, $\mu(x) \geq a$ if and only if $\mu_1(x) \geq b$; in particular, $\mu(x) = a$ if and only if $\mu_1(x) = b$, because $x$ will be a unity of $\mu_a = (\mu_1)_b$; in consequence, $\mu(x) > a$ if and only if $\mu_1(x) > b$. With this information, for each $x, y \in F$ nonzero, we have that the following equivalences are true: $\mu(x) < \mu(y)$ if and only if $\mu_1(x) < \mu_1(y)$ and $\mu(x) = \mu(y)$ if and only if $\mu_1(x) = \mu_1(y)$. We define $\psi : \mu(F) \longrightarrow \mu_1(F)$ given by

$$\psi(\mu(x)) := \mu_1(x),$$

for all $x \in F$. $\psi$ is a homomorphism of lattices, since for all $x, y \in F$, with $\mu(x) \leq \mu(y)$, we have that $\mu_1(x) \leq \mu_1(x)$ and

$$\psi(\mu(x) \vee \mu(y)) = \psi(\mu(y)) = \mu_1(y) = \mu_1(x) \vee \mu_1(y) = \psi(\mu(x)) \vee \psi(\mu(y)), \text{ and}$$

$$\psi(\mu(x) \wedge \mu(y)) = \psi(\mu(x)) = \mu_1(x) = \mu_1(x) \wedge \mu_1(y) = \psi(\mu(x)) \wedge \psi(\mu(y)).$$

Furthermore, $\psi$ is injective because, for each $x, y \in F$, if $\psi(\mu(x)) = \psi(\mu(y))$, that is $\mu_1(x) = \mu_1(y)$, then there exists $u$ a unity of $\mu_a = (\mu_1)_b$ such that $x = uy$, with which $\mu(x) = \mu(uy) = \mu(y)$. As it clear that $\psi$ is surjective, we have that $\psi : \mu(F) \longrightarrow \mu_1(F)$ is an isomorphism of lattices, and $\psi(a) = \psi(\mu(1)) = \mu_1(1) = b$. Reciprocally, we suppose that there exists an isomorphism of lattices $\psi : \mu(F) \longrightarrow \mu_1(F)$ such that maps $a$ onto $b$ and $\psi \circ \mu = \mu_1$. Then, for all $x \in F$, it is clear that $\mu(x) \geq a$ if and only if $\psi(\mu(x)) \geq \psi(a)$, that is $\mu_1(x) \geq b$. Therefore, $\mu_a = (\mu_1)_b$.

Under an analogous procedure of the proof of the Theorem 3.1, we have the following theorem of approach for $L$-valuations, knowing that it holds $(ix)$ of the Proposition 4.1, and $(iii)$ of the Proposition 4.2.

**Theorem 4.2. (Approximation Theorem for $L$-valuations)** *Let $\mu_1, \ldots, \mu_n$ be $L$-valuations not equivalent to each other of a function field $F/K$, with $n \geq 2$, and divisor elements $a_1, \ldots, a_n$ respectively, which satisfy the Archimedean property. Then, for each $x_1, \ldots, x_n \in F$ and for each $r_i \in \mu_i(F)$, with $r_i \neq \nu_i(0)$ and $i = 1, \ldots, n$, there exists $x \in F$ such that*

$$\mu_i(x - x_i) = r_i, \quad \textit{for each} \ \ i = 1, \ldots, n.$$

## 5. Examples

**1.** Let $-\infty$ and $+\infty$ be two different objects that do not belong to the set $\mathbb{Z}$ of the integer numbers, satisfying that $-\infty < n < +\infty$ for all integer number $n$. Then, $L := \mathbb{Z} \cup \{-\infty, +\infty\}$ is a complete Heyting algebra. On the other hand, let $F = K(x)$ be the rational function field over the field $K$ in the indeterminate $X$, and $f(X)$ an irreducible polynomial of $K[X]$. Then, the valuation of $F$ associated to the maximal ideal $P$ of $K[X]$ generated by $f(X)$, denoted for $\upsilon_P$, is given by $\upsilon_P(0) = +\infty$ and $\upsilon_P(\alpha(X)) = n$ if $\alpha(X) \neq 0$, $\alpha(X) = f(X)^n \dfrac{g(X)}{h(X)}$ for unique polynomials except associates $g(X), h(X) \in K[X]$ relatively prime, and for a unique $n \in \mathbb{Z}$. Then, $\varphi \circ \upsilon_P$ is an $L$-subring of valuation of $F$ with divisor element $a = 0$, where $\varphi : \mathbb{Z} \cup \{+\infty\} \to L$ is given by: $\varphi(+\infty) = +\infty$ y for each $n \in \mathbb{Z}$

$$\varphi(n) := \begin{cases} n & \text{if} \quad n \geq 0 \\ \\ -1 & \text{if} \quad n < 0. \end{cases}$$

Note that $\upsilon_P$ is an $L$-valuation of the rational funtion field $F$ with divisor element $a = 0$.

**2.** The interval $L := [0, 1]$ is a complete Heyting algebra. Again, we consider the rational function field $F = K(X)$, $f(X)$ an irreducible polynomial of $K[X]$, where each element $\alpha(X) \in F$ is expressed in the form $\alpha(X) = f(X)^n \dfrac{g(X)}{h(X)}$ for unique polynomials except associates $g(X), h(X) \in K[X]$ relatively prime, and for a unique $n \in \mathbb{Z}$. Then, the function $\nu_P : F \longrightarrow L$ given by

$$\nu_P(\alpha(X)) := \begin{cases} 1 & \text{if} \quad \alpha(X) = 0 \\ \\ 1 - \dfrac{1}{n+2} & \text{if} \quad n \geq 0 \\ \\ \dfrac{1}{4} & \text{if} \quad n < 0 \end{cases}$$

is an $L$-subring of valuation of the function field $F$. While the function $\mu_P : F \longrightarrow L$ given by

$$\mu_P(\alpha(X)) := \begin{cases} 1 & \text{if} \quad \alpha(X) = 0 \\ 1 - \dfrac{1}{n+2} & \text{if} \quad n \geq 0 \\ -\dfrac{1}{n-2} & \text{if} \quad n < 0 \end{cases}$$

is an $L-$valuation of the function field $F$.
In both cases, its divisor element is $a = 1/2$.

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